A New Simple Proof of the λ_g Conjecture and Witten Conjecture

Yi Li

Center of Mathematical Sciences, ZheJiang Univercity
Hangzhou, China, 310027

yili@cms.zju.edu.cn

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1. Introduction

- The λ_g conjecture:

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \psi^{k_1} \cdots \psi_n^{k_n} = \binom{2g+n-3}{k_1, \cdots, k_n} b_g$$

where

$$b_g := \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}, \quad g > 0; \quad b_0 = 1.$$

and B_l is the Bernoulli numbers.

- The DVV conjecture(which is equivalent to the Witten conjecture):

$$\begin{aligned}\langle \widetilde{\tau}_{b_1} \prod_{l=2}^{n} \widetilde{\tau}_{b_l} \rangle_g = & \sum_{l=2}^{n} (2b_l+1) \langle \widetilde{\tau}_{b_1+b_l-1} \prod_{k=2, k\neq l}^{n} \widetilde{\tau}_{b_k} \rangle_g + \frac{1}{2} \sum_{a+b=b_1-2} \langle \widetilde{\tau}_a \widetilde{\tau}_b \prod_{l=2}^{n} \widetilde{\tau}_{b_l} \rangle_{g-1} \\ & \frac{1}{2} \sum_{X \cup Y = \{b_2, \cdots, b_n\}} \sum_{a+b=b_1-2, g_1+g_2=g} \langle \widetilde{\tau}_a \prod_{\alpha \in X} \widetilde{\tau}_\alpha \rangle_{g_1} \langle \widetilde{\tau}_b \prod_{\beta \in Y} \widetilde{\tau}_\beta \rangle_{g_2}.\end{aligned}$$

where $\widetilde{\tau}_{b_l} = [(2b_l+1)!!] \tau_{b_l}$.

- A new closed formula of Hodge integrals:

$$\int_{\overline{\mathcal{M}}_{g,1}} \lambda_1 \lambda_g \psi_1^{2g-3} = \frac{1}{12} [g(2g-3) b_g + b_1 b_{g-1}], \quad g \geq 2.$$

1.1. λ_g Conjecture

The λ_g conjecture was first proved by Faber and Pandharipande, and their approach was to use localization technique on the stable maps to $\mathbb{P}^1$. On the other hand, Getzler and Pandharipande have showed that the λ_g conjecture is equivalent to the Virasoro conjecture for $\mathbb{P}^1$ which was a particular case of Virasoro conjecture for curve proved by Okounkov and Pandharipande in 2003.

Chiu-Chu Melissa Liu, Kefeng Liu and Jian Zhou gave a new proof by taking limits the Mariño-Vafa formula.

I.P. Goulden, D.M. Jackson and R.Vakil also gave a short proof via ELSV formula.

Before Vakil's proof, the author have just derived a simple proof of the λ_g conjecture, using the differentiable equation arising from the Mariñ-Vafa formula. As a consequence, we found two supplementary identities: one is a new closed formula of $\lambda_1\lambda_g$ integrals over moduli space $\overline{\mathcal{M}}_{g,1}$ while another identity is the recursion formula of the λ_g integrals.

Reference: Yi Li. Some Results of the Mariño-Vafa formula, *Math.Res.Lett.* **13**(2006), no.6, 847-864

1.2. Witten Conjecture

- The well-known Witten conjecture states that the intersection theory of the ψ classes on the moduli spaces of Riemann surfaces is equivalent to the "Hermitian matrix model" of two-dimensional gravity. All ψ-integrals can be efficiently computed by using the Witten conjecture, first proved by Kontsevich.
- Y.-S. Kim and Kefeng Liu gave a simple proof of the Witten conjecture by first proving a recursion formula (DVV) conjectured by Dijkgraaf-Verlinde-Verlinde, and as corollary they were able to give a simple proof of the Witten conjecture by using asymptotic analysis.
- There are other proofs such as Mirzakhani using the Weil-Petersson volume of the moduli space $\mathcal{M}_{g,n}(b)$, M.Kazarian and S.Lando using the algebro-geometric method.
- Lin Chen, Kefeng Liu and I use the method of other researchers to prove this recursion formula (DVV), therefore the Witten conjecture without using the asymptotic analysis. Combining the coefficients derived in our note and some approach, we can derive more recursion formulas of Hodge integrals.
- Reference: Lin Chen, Yi Li, and Kefeng Liu. Localization, Hurwitz Numbers and Witten Conjecture, math.AG/0609236, preprint

2. Gromov-Witten Invariants

2.1. Gromov-Witten Invariants

Let X be a projective algebraic variety (i.e., projective manifold), $\beta \in H_2(X, \mathbb{Z})$. An n**-pointed stable map** $(C, f, p_1, \cdots, p_n)$ consists of

• C connected marked curve, $f : C \to X$ morphism;

• $p_1, \cdots, p_n$ are distinct ordered smooth points of C;

• the only singularities of C are ordinary double points, or node points;

• $(C, f, p_1, \cdots, p_n)$ has only finitely many automorphisms.

Let $\overline{\mathcal{M}}_{g,n}(X, \beta)$ be the set of all equivalent class of n-pointed stable map $f : (C, p_1, \cdots, p_n) \to X$ with genus g of class β, where the equivalence relation is given by obvious identification: $(C, f, p_1, \cdots, p_n) \sim (C', f', p_1', \cdots, p_n')$ if and only if there exists a morphism $\varphi : C \to C'$ such that $f = f' \circ \varphi$.

Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford moduli stack of stable curves of genus g with n marked points. When X is smooth and $\beta = 0$, there is a simple relation between two moduli space

$$\overline{\mathcal{M}}_{g,n}(X, 0) = X \times \overline{\mathcal{M}}_{g,n}. \tag{2.1}$$

If X is smooth, then the **expected dimension** is

$$\dim\overline{\mathcal{M}}_{g,n}(X,\beta) = (1-g)(\dim X - 3) - \int_\beta K_X + n \tag{2.2}$$

$$= (1-g)(\dim X - 3) + \int_\beta c_1(T_X) + n \tag{2.3}$$

where K_X is the canonical class.

$X = \mathbb{P}^r$. We can write $\beta = dH$ with $H = c_1(\mathcal{O}_{\mathbb{P}^r}(1)) \in H^*(\mathbb{P}^r)$ (hyperplane class). Usually, this moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,\beta)$ are denoted by $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$ and the dimension are

$$\dim\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d) = rd + r + d + n - 3 - g(r-3). \tag{2.4}$$

In particular, $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r,d)$, an irreducible, normal projective variety of dimension $(r+1)d+r-3$, play an important role in mirror symmetry.(B. Lian, K. Liu and S.T. Yau, Mirror Principle I-IV)

There are two natural maps

$$\pi_1 : \overline{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow X^n, (C,f,p_1,\cdots,p_n) \longmapsto (f(p_1),\cdots,f(p_n)), \tag{2.5}$$

$$\pi_2 : \overline{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow \overline{\mathcal{M}}_{g,n}, (C,f,p_1,\cdots,p_n) \longmapsto \widetilde{C}, \tag{2.6}$$

where $\widetilde{C} \subset C$ is the stable curve given by contracting the non-stable components of C.

Let X be smooth, we have

$$\pi_1^* : H^*(X, \mathbb{Q})^{\otimes n} \longrightarrow H^*(\overline{\mathcal{M}}_{g,n}(X, \beta), \mathbb{Q}), \tag{2.7}$$

$$\pi_2^* : H_*(\overline{\mathcal{M}}_{g,n}(X, \beta), \mathbb{Q}) \longrightarrow H_*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}). \tag{2.8}$$

From the Poincare duality

$$\begin{array}{ccc} H^*(\overline{\mathcal{M}}_{g,n}(X,\beta),\mathbb{Q}) & \xrightarrow{\cong} & H_{2e-*}(\overline{\mathcal{M}}_{g,n}(X,\beta),\mathbb{Q}) \\ \downarrow & & \downarrow \\ H^{2(3g-3+n)-2e+*}(\overline{\mathcal{M}}_{g,n},\mathbb{Q}) & \xleftarrow{\cong} & H_{2e-*}(\overline{\mathcal{M}}_{g,n},\mathbb{Q}) \end{array}$$

where $e = \dim \overline{\mathcal{M}}_{g,n}(X, \beta)$, we have the **Gysin map**

$$\pi_{2!} : H^*(\overline{\mathcal{M}}_{g,n}(X, \beta), \mathbb{Q}) \longrightarrow H^{2m+*}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}), \tag{2.9}$$

where $m = (g-1)\dim X + \int_\beta \omega_X$. Define the **Gromov-Witten class**

$$I_{g,n,\beta}(\alpha_1, \cdots, \alpha_n) := \pi_{2!}(\pi_1^*(\alpha_1 \otimes \cdots \otimes \alpha_n)), \tag{2.10}$$

Define the **Gromov-Witten invariant**

$$\langle I_{g,n,\beta}\rangle(\alpha_1, \cdots, \alpha_n) := \int_{\overline{\mathcal{M}}_{g,n}} I_{g,n,\beta}(\alpha_1, \cdots, \alpha_n) \in \mathbb{Q}. \tag{2.11}$$

It is easy to see that

$$\langle I_{g,n,\beta}\rangle(\alpha_1,\cdots,\alpha_n)=\int_{\overline{\mathcal{M}}_{g,n}(X,\beta)}\pi_1^*(\alpha_1\otimes\cdots\otimes\alpha_n), \tag{2.12}$$

we denote

$$\langle\alpha_1,\cdots,\alpha_n\rangle_{g,\beta}^X:=\langle I_{g,n,\beta}\rangle(\alpha_1,\cdots,\alpha_n). \tag{2.13}$$

When X is not a smooth, the trouble arises in the fact that the algebraic stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$ may have different dimensions in different components, but we can show that there exists a **virtual fundamental class** $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}}\in H^*(\overline{\mathcal{M}}_{g,r}(X,\beta),\mathbb{Q})$ with the expected dimension.

If X is not a smooth and $n,g\geq 0$, we can also define the **Gromov-Witten invariant** $\langle I_{g,n,\beta}\rangle(\alpha_1,\cdots,\alpha_n)$ which is the rational number as following

$$\langle I_{g,n,\beta}\rangle(\alpha_1,\cdots,\alpha_n):=\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}}}\mathrm{ev}_1^*(\alpha_1)\cup\cdots\cup\mathrm{ev}_n^*(\alpha_n), \tag{2.14}$$

where the **evaluation map** ev_i are defined by

$$\mathrm{ev}_i:\overline{\mathcal{M}}_{g,n}(X,\beta)\longrightarrow X,\quad (C,f,p_1,\cdots,p_n)\longmapsto f(p_i). \tag{2.15}$$

Let X be a smooth algebraic variety, $\beta \in H_2(X, \mathbb{Z})$, there are two natural maps, evaluation map ev_i and fogetting map π_{n+1} respectively:

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \longrightarrow X, \quad (f, C, p_1, \cdots, p_n) \longmapsto f(p_i), \tag{2.16}$$

$$\pi_{n+1} : \overline{\mathcal{M}}_{g,n+1}(X, \beta) \longrightarrow \overline{\mathcal{M}}_{g,n}(X, \beta), \tag{2.17}$$

where we forget the last marked point and contract unstable components. Define the tautological section

$$s_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \longrightarrow \overline{\mathcal{M}}_{g,n+1}(X, \beta) \tag{2.18}$$

$$(f, C, p_1, \cdots, p_n) \longmapsto (f, C \cup \mathbb{P}^1, p_1, \cdots, p_i', \cdots, p_n, p_{n+1}'), \tag{2.19}$$

where $\mathbb{P}^1$ is attached to C at p_i and p_i', p_{n+1}' are distinct points of $\mathbb{P}^1$ different from the attaching point. Let ω_{n+1} be the relative dualizing sheaf of π_{n+1} (cf, Hartshorne, *Algebraic Geometry*, 214), denote

$$\mathbb{L}_i := s_i^* \omega_{n+1}$$

be the line bundle over $\overline{\mathcal{M}}_{g,n}(X, \beta)$ whose fiber over $(f, C, p_1, \cdots, p_n) \in \overline{\mathcal{M}}_{g,n}(X, \beta)$ is the cotangent line $T^*_{x_i}C$ at the i-th marked point p_i. Let $\psi_i = c_1(\mathbb{L}_i)$ be the first Chern class of $\mathbb{L}_i$.

2.2. Gravitational Correlator

Given classes $\gamma_1, \cdots, \gamma_n \in H^*(X, \mathbb{Q})$ and nonnegtive integers k_i for each $i = 1, \cdots, n$, define a **gravitational correlator**

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_{g,\beta}^X := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\text{virt}}} \prod_{i=1}^n \left(\psi^{k_i} \cup \text{ev}_i^*(\gamma_i) \right), \qquad (2.20)$$

consider the generating function

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_g^X := \sum_{0 \neq \beta \in H_2(X,\mathbb{Z})} q^\beta \langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_{g,\beta}^X, \qquad (2.21)$$

where $q^\beta = e^{2\pi\sqrt{-1}\int_\beta \omega}$ as usual. Introduce variables t_k^a, $k \geq 0, 0 \leq a \leq n$ such that $t_0^a = t_a$. The **genus g gravitational Gromov-Witten potential** is given by

$$\langle\langle \ \rangle\rangle_g^X := \sum_{n=0}^{+\infty} \frac{1}{n!} \sum_{k_1,\cdots,k_n; a_1,\cdots,a_n} t_{k_n}^{a_n} \cdots t_{k_1}^{a_1} \langle \tau_{k_1,a_1} \cdots \tau_{k_n,a_n} \rangle_g^X, \qquad (2.22)$$

where $\tau_{k,a}$ is an abbreviation for $\tau_k(\gamma_a)$. The **total Gromov-Witten potential** is

$$Z(X) := \exp\left(\sum_{g \geq 0} \hbar^{g-1} \langle\langle \ \rangle\rangle_g^X \right), \qquad (2.23)$$

where $\hbar$ is a parameter. Furthermore, we define the **genus g couplings**

$$\langle\langle \tau_{k_1,a_1} \cdots \tau_{k_n,a_n} \rangle\rangle := \partial_{k_1,a_1} \cdots \partial_{k_n,a_n} \langle\langle\ \rangle\rangle_g^X, \tag{2.24}$$

where $\partial_{m,a} := \partial/\partial t_m^a$.

String Equation

If $n + 2g \geq 4$ or $\beta \neq 0$ and $n \geq 1$, then

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_{n-1}}(\gamma_{n-1}) \tau_0(1) \rangle_{g,\beta}^X \tag{2.25}$$

$$= \sum_{i=1}^{n-1} \langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_{i-1}}(\gamma_{i-1}) \tau_{k_i - 1}(\gamma_i) \tau_{k_{i+1}}(\gamma_{i+1}) \cdots \tau_{k_{n-1}}(\gamma_{n-1}) \rangle_{g,\beta}^X \tag{2.26}$$

If X is a point and $\beta = 0$, then

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{k_1} \cdots \psi_n^{k_n} = \sum_{i=1}^{n} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_i^{k_i - 1} \cdots \psi_n^{k_n}. \tag{2.27}$$

In particular, if $k_1 + \cdots + k_n = n - 3$, then

$$\int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} = \binom{n-3}{k_1, \cdots, k_n}. \tag{2.28}$$

Dilaton Equation

If $2g-2+n>0$, we have

$$\langle \tau_1 \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle^X_{g,\beta} = (2g-2+n)\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\rangle^X_{g,\beta}. \qquad (2.29)$$

When X is a point and $\beta=0$, we have the dilaton equation for $\overline{\mathcal{M}}_{g,n}$:

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{k_1}\cdots\psi_n^{k_n}\psi_{n+1} = (2g-2+n)\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\cdots\psi_n^{k_n}. \qquad (2.30)$$

Example

Let $X=\mathbb{P}^1$, $\beta=d[l]$, $l\subset\mathbb{P}^1$ is a line, $\dim[\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1,d)]^{\mathrm{virt}} = \dim\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1,d) = 2d+n-2$. Since $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^1,1)\cong\mathbb{P}^1$, the cotangent line bundle $\mathbb{L}\cong T^*_{\mathbb{P}^1}$. There are some examples:

$$\langle\tau_1\rangle^{\mathbb{P}^1}_{0,1} = \int_{\mathbb{P}^1} c_1(T^*_{\mathbb{P}^1}) = -\int_{\mathbb{P}^1} c_1(T_{\mathbb{P}^1}) = -2,$$

Let $H=c_1(\mathcal{O}_{\mathbb{P}^1}(1))$, then $\langle H\rangle^{\mathbb{P}^1}_{0,1}=1$, using the dilaton equation, we have

$$\langle H,\tau_1\rangle^X_{0,1} = (2\times 0-2+1)\langle H\rangle^{\mathbb{P}^1}_{0,1} = -\langle H\rangle^{\mathbb{P}^1}_{0,1} = -1.$$

3. Hodge Integrals

3.1. Hodge Integrals

Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford moduli stack of stable curves of genus g with n marked points. Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the universal curve, and let ω_π be the relative dualizing sheaf. The Hodge bundle

$$\mathbb{E} := \pi_* \omega_\pi$$

is a rank g vector bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over $[(C, x_1, \cdots, x_n)] \in \overline{\mathcal{M}}_{g,n}$ is $H^0(C, \mathcal{O}_C)$, the complex vector space of holomorphic one forms on C. Let $s_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ denote the section of π which corresponds to the i-th marked point, and let

$$\mathbb{L}_i := s_i^* \omega_\pi$$

be the line bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over $[(C, x_1, \cdots, x_n)] \in \overline{\mathcal{M}}_{g,n}$ is the cotangent line $T^*_{x_i} C$ at the i-th marked point x_i. Consider the **Hodge integral**

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{j_1} \cdots \psi_n^{j_n} \lambda_1^{k_1} \cdots \lambda_g^{k_g} \tag{3.1}$$

where $\psi_i = c_1(\mathbb{L}_i)$ is the first Chern class of $\mathbb{L}_i$, and $\lambda_j = c_j(\mathbb{E})$ is the j-th Chern class of $\mathbb{E}$. The dimension of $\overline{\mathcal{M}}_{g,n}$ is $3g-3+n$, hence (3.1) is equal to zero unless $\sum_{i=1}^n j_i + \sum_{i=1}^g ik_i = 3g-3+n$. Let

$$\Lambda_g^\vee(u) := u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g = \sum_{i=0}^{g} (-1)^i \lambda_i u^{g-i} \tag{3.2}$$

be the Chern polynomial of the dual bundle $\mathbb{E}^\vee$ of $\mathbb{E}$.

3.2. Mumford's relations

Let $c_t(\mathbb{E}) := \sum_{i=0}^g t^i \lambda_i$, then we have $c_{-t}(\mathbb{E}) = t^g \Lambda_g^\vee\left(\frac{1}{t}\right)$. **Mumford's relations** state that $c_t(\mathbb{E})c_{-t}(\mathbb{E}) = 1$ or $\Lambda_g^\vee(t)\Lambda_g^\vee(-t) = (-1)^g t^{2g}$, then $\lambda_k^2 = \sum_{i=1}^k (-1)^{i+1} 2\lambda_{k-i}\lambda_{k+i}$ where $\lambda_0 = 1$ and $\lambda_k = 0$ for $k > g$. Let $x_1, \cdots, x_g$ be the formal Chern roots of $\mathbb{E}$, the Chern character is defined by

$$\mathrm{ch}(\mathbb{E}) := \sum_{i=1}^{g} e^{x_i} = g + \sum_{n=1}^{+\infty} \sum_{i=1}^{g} \frac{x_i^n}{n!} := \mathrm{ch}_0(\mathbb{E}) + \sum_{n=1}^{2g} \mathrm{ch}_n(\mathbb{E}).$$

From the above identities we have the relation between $\mathrm{ch}_n(\mathbb{E})$ and λ_n:

$$n!\mathrm{ch}_n(\mathbb{E}) = \sum_{i+j=n} (-1)^{i-1} i\lambda_i\lambda_j, \quad n < 2g; \quad \mathrm{ch}_n(\mathbb{E}) = 0, \quad n \geq 2g. \tag{3.3}$$

3.3. Witten Conjecture/Kontsevich Theorem

All ψ-integrals can be efficiently computed by using Witten conjecture, proved by Kontsevich. For convenience, we use Witten's notation

$$\langle \tau_{\beta_1} \cdots \tau_{\beta_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{j_1} \cdots \psi_n^{j_n}. \tag{3.4}$$

The natural generating function for the ψ-integrals described above is

$$F_g(t) := \sum_{n\geq 0} \frac{1}{n!} \sum_{k_1,\cdots,k_n} t_{k_1} \cdots t_{k_n} \langle \tau_{\beta_1} \cdots \tau_{\beta_n} \rangle_g, \quad F(t,\lambda) := \sum_{g\geq 0} F_g \lambda^{2g-2}. \tag{3.5}$$

The first system of differential equations conjectured by Witten are the KDV equations. Let $F(t) := F(t,1)$, define

$$\langle\langle \tau_{\beta_1} \cdots \tau_{\beta_n} \rangle\rangle := \frac{\partial}{\partial t_{k_1}} \cdots \frac{\partial}{\partial t_{k_n}} F(t), \tag{3.6}$$

then the KDV equations for $F(t)$ are equivalent to the set of equations for $n \geq 1$:

$$(2n+1)\langle\langle \tau_n \tau_0^2 \rangle\rangle = \langle\langle \tau_{n-1}\tau_0 \rangle\rangle \langle\langle \tau_0^3 \rangle\rangle + 2\langle\langle \tau_{n-1}\tau_0^2 \rangle\rangle \langle\langle \tau_0^2 \rangle\rangle + \frac{1}{4}\langle\langle \tau_{n-1}\tau_0^4 \rangle\rangle. \tag{3.7}$$

We have the **point Virasoro theorem** $L_n e^{F(t,\lambda)} = 0, \quad n \geq -1.$

$$L_k = \sum_{m=0}^{+\infty} \frac{\Gamma(k+m+\frac{3}{2})}{\Gamma(m+\frac{1}{2})}(t_m - \delta_{m,1})\partial_{m+k} + \frac{\hbar}{2}\sum_{m=0}^{k-1}(-1)^{m+1}\frac{\Gamma(k-m+\frac{1}{2})}{\Gamma(-m-\frac{1}{2})}\partial_m\partial_{k-m-1},$$

$$L_{-1} = \sum_{m=0}^{+\infty}(t_m - \delta_{m,1})\partial_{m-1} + \frac{1}{2\hbar}t_0^2, \quad L_0 = \sum_{m=0}^{+\infty}(m+\frac{1}{2})(t_m - \delta_{m,1})\partial_m + \frac{1}{16},$$

where $\hbar = \lambda^2$ and $\partial_i = \partial/\partial t_i$.

An **Virasoro algebra** is a sequence of differential operators $\{L_j\}_{-\infty}^{+\infty}$ such that $[L_k, L_l] = (k-l)L_{k+l}$. Witten's conjecture is the special case of the well-known **Virosoro Conjecture** which says that if X is a smooth projective variety, then there is a Virasoro algebra $\{L_j\}$ of formal differential operator in the t_d^j such that $L_j Z(X) = 0$ for all $k \geq -1$.

Another famous conjecture on Hodge integrals, which is not equivalent to Witten conjecture, is Faber conjecture. The initial step was first proved by C.Faber using Witten conjecture. Goulden, Jackson and Vakil have given a proof of the Faber conjecture up to three points. As note by Kefeng Liu and Hao Xu (math.AG/0609367), it is very clear to have a simpler and direct explanation.

If X is a point, then the following notation

$$Z := Z(X) = \exp\left(\sum_{g\geq 0}\hbar^{g-1}\sum_{n\geq 0}\frac{1}{n!}\sum_{k_1,\cdots,k_n} t_{k_1}\cdots t_{k_n}\langle\tau_{k_1}\cdots\tau_{k_n}\rangle_g\right), \qquad (3.8)$$

where

$$\langle\tau_{k_1}\cdots\tau_{k_n}\rangle_g := \int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\cdots\psi_n^{k_n}, \qquad (3.9)$$

is coincided with $e^{F(t,\lambda)}$, $\lambda^2 = \hbar$.

Let E_1 and E_2 be vector bundles on M, and let pr_1 and pr_2 be the projection from $M \times M$ onto the first and second factor M respectively.We define the **external tensor product** of E_1 and E_2 as following

$$E_1 \boxtimes E_2 := \mathrm{pr}_1^* E_1 \otimes \mathrm{pr}_2^* E_2. \qquad (3.10)$$

If X is smooth, from the isomorphism $\overline{\mathcal{M}}_{g,n}(X,0) \cong X \times \overline{\mathcal{M}}_{g,n}$, the virtual fundamental class in K-group is given by

$$[\overline{\mathcal{M}}_{g,n}(X,0)]^{\mathrm{virt}} = e(T_X \boxtimes \mathbb{E}^\vee) \cap [X \times \overline{\mathcal{M}}_{g,n}], \qquad (3.11)$$

where $e(T_X \boxtimes \mathbb{E}^\vee)$ is the top Chern class of $T_X \boxtimes \mathbb{E}^\vee$.

Let $\{\gamma_a\}$ be a basis of $H^*(X,\mathbb{Q})$, then

$$\begin{aligned}\langle\tau_{k_1}(\gamma_{a_1})\cdots\tau_{k_n}(\gamma_{a_n})\rangle_{g,0}^X &= \int_{[\overline{\mathcal{M}}_{g,n}(X,0)]^{\mathrm{virt}}}\gamma_{a_1}\cdots\gamma_{a_n}\psi_1^{k_1}\cdots\psi_n^{k_n}\\ &= \int_{X\times\overline{\mathcal{M}}_{g,n}}\gamma_{a_1}\cdots\gamma_{a_n}\psi_1^{k_1}\cdots\psi_n^{k_n}\cup e(T_X\boxtimes\mathbb{E}^\vee).\end{aligned}$$

X is a curve. T_X is a line bundle, and

$$c(T_X\boxtimes\mathbb{E}^\vee)=\sum_{i=0}^g(1+c_1(T_X))^i c_{g-i}(\mathbb{E}^\vee),\quad e(T_X\boxtimes\mathbb{E}^\vee)=\sum_{i=0}^g c_1(T_X)^i c_{g-i}(\mathbb{E}^\vee).$$

Since $c_1(T_X)^i=0, i>1$. we have

$$e(T_X\boxtimes\mathbb{E}^\vee)=(-1)^g\lambda_g+(-1)^{g-1}c_1(T_X)\lambda_{g-1}. \tag{3.12}$$

X is a surface.

$$e(T_X\boxtimes\mathbb{E}^\vee)=-c_1(X)\lambda_g\lambda_{g-1}+c_1(X)^2\lambda_g\lambda_{g-2}. \tag{3.13}$$

X is a threefold.

$$e(T_x\boxtimes\mathbb{E}^\vee)=\frac{(-1)^g}{2}[c_3(X)-c_2(X)c_1(X)]\lambda_{g-1}^3. \tag{3.14}$$

- X is a curve.

$$\langle \tau_{k_1}(\gamma_{a_1})\cdots\tau_{k_n}(\gamma_{a_n})\rangle^X_{g,0} = (-1)^g\left(\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\cdots\psi_n^{k_n}\lambda_g\right)\int_X \gamma_{a_1}\cdots\gamma_{a_n} + (-1)^{g-1}\left(\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\cdots\psi_n^{k_n}\lambda_{g-1}\right)\int_X \gamma_{a_1}\cdots\gamma_{a_n}c_1(T_X)$$

- X is a surface.

$$\langle \tau_{k_1}(\gamma_{a_1})\cdots\tau_{k_n}(\gamma_{a_n})\rangle^X_{g,0} = -\left(\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\cdots\psi_n^{k_n}\lambda_g\lambda_{g-1}\right)\int_X \gamma_{a_1}\cdots\gamma_{a_n}c_1(T_X) + \left(\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\cdots\psi_n^{k_n}\lambda_g\lambda_{g-2}\right)\int_X \gamma_{a_1}\cdots\gamma_{a_n}c_1(T_X)^2$$

- X is a threefold.

$$\langle \tau_{k_1}(\gamma_{a_1})\cdots\tau_{k_n}(\gamma_{a_n})\rangle^X_{g,0} = \frac{(-1)^g}{2}\left(\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\cdots\psi_n^{k_n}\lambda_{g-1}^3\right)\int_X \gamma_{a_1}\cdots\gamma_{a_n}[c_3(T_X)-c_2(T_X)c_1(T_X)]$$

4. Mariño-Vafa Formula

4.1. Geometric side of Mariño-Vafa Formula

A partition of a positive integer d is a sequence of integers $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{l(u)} > 0$ such that $\mu_1 + \cdots + \mu_{l(\mu)} = d = |\mu|$. For every partition μ, define

$$\begin{aligned} \mathcal{C}_{g,\mu}(\tau) &:= -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|\mathrm{Aut}(\mu)|}[\tau(\tau+1)]^{l(\mu)-1}\prod_{i=1}^{l(\mu)}\frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!} \quad (4.1) \\ &\cdot \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} \\ \mathcal{C}_\mu(\lambda;\tau) &:= \sum_{g\geq 0}\lambda^{2g-2+l(\mu)}\mathcal{C}_{g,\mu}(\tau). \quad (4.2) \end{aligned}$$

We introduce formal variables $p = (p_1, p_2, \cdots, p_n, \cdots)$ and define $p_\mu = p_{\mu_1}\cdots p_{\mu_{l(\mu)}}$ for a partition μ. Define generating functions

$$\mathcal{C}(\lambda;\tau,p) := \sum_{|\mu|\geq 0}\mathcal{C}_\mu(\lambda;\tau)p_\mu \quad (4.3)$$

$$\mathcal{C}(\lambda;\tau,p)^\bullet := e^{\mathcal{C}(\lambda;\tau,p)} \quad (4.4)$$

4.2. Representation side of Mariño-Vafa Formula

For a partition μ, let $m_i(\mu) = |\{j|\mu_j = i, 1 \leq j \leq l(\mu)\}|$. The automorphism group $\mathrm{Aut}(\mu)$ of μ consists of possible permutations among the μ_i's, hence its order is given by $|\mathrm{Aut}(\mu)| = \prod_i m_i(\mu)!$, define the numbers

$$\kappa_\mu := \sum_{i=1}^{l(\mu)} \mu_i(\mu_i - 2i + 1), \quad z_\mu := \prod_j m_j(\mu)! j^{m_j(\mu)}, \quad h(x) := \mu_i + \mu'_j - i - j + 1.$$

where μ' is the conjugate of partition μ. Each partition μ of d corresponds to a conjugacy class $C(\mu)$ of the symmetric group S_d and each partition ν corresponds to an irreducible representation R_ν of S_d, let $\chi_\nu(C(\mu)) := \chi_{R_\nu}(C(\mu))$ be the value of the character χ_{R_ν} on the conjugacy class $C(\mu)$. Denote

$$\mathcal{W}_\mu(q) := \prod_{1 \leq i < j \leq l(\mu)} \frac{[\mu_i - \mu_j + j - i]}{[j - i]} \prod_{i=1}^{l(\mu)} \frac{\prod_{v=-i+1}^{\mu_i - i} q^{v/2}}{\prod_{v=1}^{\mu_i} [v - i + l(\mu)]}, \quad (4.5)$$

$$R(\lambda; \tau, p)^\bullet := \sum_\mu \frac{\chi_\nu(C(\mu))}{z_\mu} e^{\sqrt{-1}(\tau + 1/2)\kappa_\nu \lambda/2} \mathcal{W}_\nu(q) p_\mu \quad (4.6)$$

where $[x] := q^{x/2} - q^{-x/2}$ and $q := e^{\sqrt{-1}\lambda}$.

4.3. Mariño-Vafa Formula

2003, Chiu-chu Melissa Liu, Kefeng Liu and Jian Zhou (Liu-Liu-Zhou. *A proof of a conjecture of Mariño-Vafa on Hodge Integrals*, J. Differential Geom. **65** (2003), 289-340.) have proved the following formula which was conjectured by Mariño and Vafa.

$$\mathcal{C}(\lambda;\tau,p)^\bullet = R(\lambda;\tau,p)^\bullet. \tag{4.7}$$

In physics, the left-hand side comes from 2D quantum gravity and the right-hand side comes from 2D Yang-Mills theory. One can in principle compute almost Hodge integrals, except the ones(Wen-Xuan Lu, Science in China Ser. A Math., 2005, 35(11): 1276-1287.).

The Mariño-Vafa formula can be generalized to two partition, three-partition and so on. For two-partition, the relative formula is

$$G^\bullet(\lambda;p^+,p^-;\tau) = R^\bullet(\lambda;p^+,p^-;\tau) \tag{4.8}$$

where $p^\pm = (p_1^\pm, p_2^\pm, \cdots)$ are formal variables. The generating function $R^\bullet(\lambda;p^+,p^-;\tau)$ is a combinational expression involving the representation theory of Kac-Moody Lie algebras. It is related to the HOMFLY polynomial of the Hopf link and the Chern-Simon theory.

5. A simple proof of the λ_g Conjecture

Introduce the notation (B_l is the Bernoulli numbers)

$$b_g := \frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}, \quad g>0; \quad b_0 = 1. \tag{5.1}$$

5.1. The simplest case: n=1.

Liu-Liu-Zhou (*Mariño-Vafa formula and Hodge integral identities*, J. Algebraic Geom. **15** (2006), 379-398.) have showed the following consequences:

- Given a simple proofs of the λ_g conjecture:

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\cdots\psi_n^{k_n}\lambda_g = \binom{2g+n-3}{k_1,\cdots,k_n} b_g. \tag{5.2}$$

- Recompute the following closed formula for Hodge integrals:

$$\int_{\overline{\mathcal{M}}_{g,0}} \lambda_{g-2}\lambda_{g-1}\lambda_g = \frac{1}{2(2g-2)!}\frac{|B_{2g-2}|}{2g-2}\frac{|B_{2g}|}{2g}, \tag{5.3}$$

$$\int_{\overline{\mathcal{M}}_{g,1}} \frac{\lambda_{g-1}}{1-\psi_1} = b_g\sum_{i=1}^{2g-1}\frac{1}{i} - \frac{1}{2}\sum_{g_1+g_2=g, g_1,g_2>0}\frac{(2g_1-1)!(2g_2-1)!}{(2g-1)!}b_{g_1}b_{g_2}.$$

We follow Liu-Liu-Zhou's method to derive some new Hodge integral identities (where $1 \leq m \leq 2g-3$ and $g \geq 2$):

$$
\begin{aligned}
& -(2g-2-m)!(-1)^{2g-3-m} \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \mathrm{ch}_{2g-2-m}(\mathbb{E}) \psi_1^m \\
= \;& b_g \sum_{k=0}^{m-1} \frac{(-1)^{2g-1-k}}{2g-1-k} \binom{2g-1}{k} \binom{2g-1-k}{2g-1-m} B_{2g-1-m} \\
+ \;& \frac{1}{2} \sum_{g_1+g_2=g, g_1,g_2>0} b_{g_1} b_{g_2} \sum_{k=0}^{\min(2g_2-1, m-1)} \frac{(-1)^{2g_2-1-k}}{2g-1-k} \binom{2g_2-1}{k} \binom{2g-1-k}{2g-1-m} B_{2g-1-m}.
\end{aligned}
$$

5.2. A new Closed Formula for Hodge integral

As a consequence, we find a new Hodge integral identity: If $g \geq 2$, then

$$
\int_{\overline{\mathcal{M}}_{g,1}} \lambda_1 \lambda_g \psi_1^{2g-3} = \frac{1}{12} [g(2g-3)b_g + b_1 b_{g-1}] \tag{5.4}
$$

5.3. Another Simple Proof of λ_g Conjecture

Let $|\mu| = d, l(\mu) = n$, denote by $[\mathcal{C}_{g,\mu}(\tau)]_k$ the coefficient of τ^k in the polynomial $\mathcal{C}_{g,\mu}(\tau)$, and let

$$J^0_{g,\mu}(\tau) := \sqrt{-1}^{|\mu|-l(\mu)}\mathcal{C}_{g,\mu}(\tau) \tag{5.5}$$

$$J^1_{g,\mu}(\tau) := \sqrt{-1}^{|\mu|-l(\mu)-1}\Bigg(\sum_{\nu\in J(\mu)} I_1(\nu)\mathcal{C}_{g,\nu}(\tau) + \sum_{\nu\in C(\mu)} I_2(\nu)\mathcal{C}_{g-1,\nu}(\tau) + \sum_{g_1+g_2=g,\nu^1\cup\nu^2\in C(\mu)} I_3(\nu^1,\nu^2)\mathcal{C}_{g_1,\nu^1}(\tau)\mathcal{C}_{g_2,\nu^2}(\tau)\Bigg). \tag{5.6}$$

The set $J(\mu)$ consists of partitions of d of the form

$$\nu = (\mu_1,\cdots,\widehat{\mu}_i,\cdots,\widehat{\mu}_j,\mu_{l(\mu)},\mu_i+\mu_j),\quad I_1(\nu) = \frac{\mu_i+\mu_j}{1+\delta^{\mu_i}_{\mu_j}} m_{\mu_i+\mu_j}(\nu)$$

and the set $C(\mu)$ consists of partitions of d of the form

$$\nu = (\mu_1,\cdots,\widehat{\mu}_i,\cdots,\mu_{l(\mu)},j,k)$$

where $j+k=\mu_i$. Liu-Liu-Zhou (Liu-Liu-Zhou. *A proof of a conjecture of Mariño-Vafa on Hodge Integrals*, J. Differential Geom. **65** (2003), 289-340.) have proved the following differential equation:

$$\frac{d}{d\tau}J^0_{g,\mu}(\tau) = -J^1_{g,\mu}(\tau). \tag{5.7}$$

It is straightforward to check that

$$[\mathcal{C}_{g,\mu}(\tau)]_{n-1} = -\frac{\sqrt{-1}^{d+n}}{|\mathrm{Aut}(\mu)|}\int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^{n}(1-\mu_i\psi_i)},$$

$$\left[\sum_{\nu\in J(\mu)} I_1(\nu)\mathcal{C}_{g,\nu}(\tau)\right]_{n-2} = -\frac{\sqrt{-1}^{d+n-1}}{|\mathrm{Aut}(\nu)|}\int_{\overline{\mathcal{M}}_{g,n-1}} \frac{\lambda_g}{\prod_{i=1}^{n-1}(1-\mu_i\psi_i)},$$

$$\left[\sum_{\nu\in C(\mu)} I_2(\nu)\mathcal{C}_{g-1,\nu}(\tau)\right]_{n-2} = \left[\sum_{g_1+g_2=g,\nu^1\cup\nu^2\in C(\mu)} I_3(\nu^1,\nu^2)\mathcal{C}_{g_1,\nu^1}(\tau)\mathcal{C}_{g_2,\nu^2}(\tau)\right]_{n-2} = 0,$$

hence, we have the identity

$$\frac{n-1}{|\mathrm{Aut}(\mu)|}\int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^{n}(1-\mu_i\psi_i)} = \sum_{\nu\in J(\mu)} \frac{I_1(\nu)}{|\mathrm{Aut}(\nu)|}\int_{\overline{\mathcal{M}}_{g,n-1}} \frac{\lambda_g}{\prod_{i=1}^{n-1}(1-\nu_i\psi_i)}. \tag{5.8}$$

Theorem: For any partition $\mu:\mu_1\geq\mu_2\geq\cdots\geq\mu_n>0$ of d and $g>0$, then

$$\int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^{n}(1-\mu_i\psi_i)} = d^{2g+n-3}b_g \Longrightarrow \int_{\overline{\mathcal{M}}_{g,n}} \lambda_g\prod_{l=1}^{n}\psi_l^{k_l} = \binom{2g+n-3}{k_1,\cdots,k_n}b_g, \tag{5.9}$$

5.4. A Recursion Formula of the λ_{g-1} Integral

Ezra Getzler, A.Okounkov and R.Pandharipande have derived explicit formula for the multipoint series of $\mathbb{P}^1$ in degree 0 from the Toda hierarchy, then they obtained **certain formulas** for the Hodge integrals $\int_{\overline{\mathcal{M}}_{g,n}} \lambda_{g-1}\psi_1^{k_1}\cdots\psi_n^{k_n}$. In this section we give an **effective recursion formula** of the λ_{g-1} integrals using Mariño-Vafa formula. Now, we can state our main theorem in this section.

Theorem: For any partition μ with $l(\mu)=n$, we have the following recursion formula

$$\begin{aligned}
&\frac{n}{|\mathrm{Aut}(\mu)|}\left[n-1+\sum_{i=1}^{n}\sum_{a=1}^{\mu_i-1}\frac{\mu_i}{a}\right]\int_{\overline{\mathcal{M}}_{g,n}}\frac{\lambda_g}{\prod_{i=1}^{n}(1-\mu_i\psi_i)}\\
&-\frac{n}{|\mathrm{Aut}(\mu)|}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\lambda_{g-1}+\sum_{k\geq 0}k!(-1)^{k-1}\mathrm{ch}_k(\mathbb{E})\lambda_g}{\prod_{i=1}^{n}(1-\mu_i\psi_i)}\\
&=\sum_{\nu\in J(\mu)}\frac{I_1(\nu)}{|\mathrm{Aut}(\nu)|}\left[n-2+\sum_{i=1}^{n-1}\sum_{a=1}^{\nu_i-1}\frac{\nu_i}{a}\right]\int_{\overline{\mathcal{M}}_{g,n-1}}\frac{\lambda_g}{\prod_{i=1}^{n-1}(1-\nu_i\psi_i)}\\
&-\sum_{\nu\in J(\mu)}\frac{I_1(\nu)}{|\mathrm{Aut}(\nu)|}\int_{\overline{\mathcal{M}}_{g,n-1}}\frac{\lambda_{g-1}+\sum_{k\geq 0}k!(-1)^{k-1}\mathrm{ch}_k(\mathbb{E})\lambda_g}{\prod_{i=1}^{n-1}(1-\nu_i\psi_i)}\\
&+\sum_{g_1+g_2=g,g_1,g_2\geq 0}\ \sum_{\nu^1\cup\nu^2\in C(\mu)}\frac{I_3(\nu^1,\nu^2)}{|\mathrm{Aut}(\nu^1)||\mathrm{Aut}(\nu^2)|}\int_{\overline{\mathcal{M}}_{g_1,n_1}}\frac{\lambda_{g_1}}{\prod_{i=1}^{n_1}(1-\nu_i^1\psi_i)}\int_{\overline{\mathcal{M}}_{g_2,n_2}}\frac{\lambda_{g_2}}{\prod_{i=1}^{n_2}(1-\nu_i^2\psi_i)}.
\end{aligned}$$

In this subsection, we re-derive the λ_g-integral from the theorem. Let $\mu_i = Nx_i$ for some $N \in \mathbb{N}$ and $x_i \in \mathbb{R}$, from Kim-Liu's (Y.-S. Kim, K. Liu. *A simple proof of Witten conjecture through localization*, math.AG/0508384.) method and consider the coefficients of $\ln N N^{2g+n-2}$, then

$$
\begin{aligned}
& n(x_1+\cdots+x_n)\prod_{l=1}^{n} x_l^{k_l}\int_{\overline{\mathcal{M}}_{g,n}}\lambda_g\prod_{l=1}^{n}\psi_l^{k_l} \\
= & \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}(x_i+x_j)^{k_i+k_j}(x_1+\cdots+x_n)\prod_{l\neq i,j}x_l^{k_l}\int_{\overline{\mathcal{M}}_{g,n-1}}\lambda_g\psi^{k_i+k_j-1}\prod_{l\neq i,j}\psi_l^{k_l} \\
+ & (x_1+\cdots+x_n)\prod_{l=1}^{n} x_l^{k_l}\int_{\overline{\mathcal{M}}_{g,n}}\lambda_g\prod_{l=1}^{n}\psi_l^{k_l},
\end{aligned}
$$

i.e.

$$
(n-1)\prod_{l=1}^{n} x_l^{k_l}\int_{\overline{\mathcal{M}}_{g,n}}\lambda_g\prod_{l=1}^{n}\psi_l^{k_l} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}(x_i+x_j)^{k_i+k_j}\prod_{l\neq i,j}x_l^{k_l}\int_{\overline{\mathcal{M}}_{g,n-1}}\lambda_g\psi^{k_i+k_j-1}\prod_{l\neq i,j}\psi_l^{k_l}.
$$

After introducing the formal variables $s_l \in \mathbb{R}^+$ and applying the Laplace transformation

$$
\int_0^{+\infty} x^k e^{-x/2s}dx = k!(2s)^{k+1}, \quad s>0,
$$

we select the coefficient of $\prod_{l=1}^{n}(2s_l)^{k_l+1}$ from the transformation, then we derive

$$(n-1)\int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \prod_{l=1}^{n} \psi_l^{k_l} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i} \frac{(k_i+k_j)!}{k_i!k_j!} \int_{\overline{\mathcal{M}}_{g,n-1}} \lambda_g \psi^{k_i+k_j-1} \prod_{l\neq i,j} \psi_l^{k_l}. \tag{5.10}$$

By the induction of n, we obtain the λ_g conjecture

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \prod_{l=1}^{n} \psi_l^{k_l} = \binom{2g+n-3}{k_1,\cdots,k_n} b_g,$$

in fact, in (5.10) we have

$$\begin{aligned} RHS &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i} \frac{(k_i+k_j)!}{k_i!k_j!} \frac{(2g+n-4)!}{\prod_{l\neq i,j} k_l!(k_i+k_j-1)!} b_g \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i} \frac{k_i+k_j}{2g+n-3} \binom{2g+n-3}{k_1,\cdots,k_n} b_g, \end{aligned}$$

note that $k_1+\cdots+k_n = 2g+n-3$, therefore

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}(k_i+k_j) = 2(n-1)(2g+n-3).$$

We have found the **singular part** $\sum_{i=1}^{n}\sum_{a=1}^{\mu_i-1}\frac{\mu_i}{a}$ in above theorem, using the following theorem, we can eliminate this part and derive the recursion formula of λ_{g-1}-integral. The notation $[F]_{sing}$ means the singular part of F.

First, we have

$$\left[\frac{LHS}{d^{2g+n-4}b_g}\right]_{sing} = n\sum_{i=1}^{n}\sum_{a=1}^{\mu_i-1}\frac{\mu_i}{a}d,$$

$$\left[\frac{RHS}{d^{2g+n-4}b_g}\right]_{sing} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}(\mu_i+\mu_j)\left[\sum_{l\neq i,j}\sum_{a=1}^{\mu_l-1}\frac{\mu_l}{a}+(\mu_i+\mu_j)\sum_{a=1}^{\mu_i+\mu_j-1}\frac{1}{a}\right] + \sum_{i=1}^{n}\sum_{a=1}^{\mu_i-1}\frac{\mu_i}{a}d - \sum_{i=1}^{n}\sum_{j\neq i}\sum_{a=\mu_j+1}^{\mu_i+\mu_j-1}\frac{\mu_j(\mu_i+\mu_j)}{a}.$$

Theorem Under the above notation, we have

$$\left[\frac{RHS}{d^{2g+n-4}b_g}\right]_{sing} = \left[\frac{LHS}{d^{2g+n-4}b_g}\right]_{sing} + 2(n-1)d.$$

Let $\mathbb{R}^k[\mu_1,\cdots,\mu_n]$ be the space of all homogeneous polynomials with real coefficients in $\mu_1,\cdots,\mu_n$ of degree k, then it is the subring of $\mathbb{R}[\mu_1,\cdots,\mu_n]$. From the theorem , we obtain the recursion formula of λ_{g-1} Hodge integral.

Theorem: For any partition μ with $l(\mu)=n$ and $|\mu|=d$, we have the recursion formula

$$\begin{aligned}&\frac{n}{|\mathrm{Aut}(\mu)|}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\lambda_{g-1}}{\prod_{i=1}^{n}(1-\mu_i\psi_i)}\\ =&\sum_{\nu\in J(\mu)}\frac{I_1(\nu)}{|\mathrm{Aut}(\nu)|}\int_{\overline{\mathcal{M}}_{g,n-1}}\frac{\lambda_{g-1}}{\prod_{i=1}^{n-1}(1-\nu_i\psi_i)}\\ -&\sum_{g_1+g_2=g,\nu^1\cup\nu^2\in C(\mu)}\frac{I_3(\nu^1,\nu^2)}{|\mathrm{Aut}(\nu^1)||\mathrm{Aut}(\nu^2)|}d_1^{2g_1+n_1-3}d_2^{2g_2+n_2-3}b_{g_1}b_{g_2}.\end{aligned}$$

under the ring $\mathbb{R}^{2g-2+n}[\mu_1,\cdots,\mu_n]$, where $l(\nu^i)=n_i$ and $|\nu^i|=d_i$ for $i=1,2$.

When we consider the simplest case $n=1$, the above identity become the formula used in [Liu-Liu-Zhou].

5.5. Some Examples of The Main Theorem

$$\int_{\overline{\mathcal{M}}_{3,1}}\lambda_1\lambda_2\lambda_3\psi_1=\frac{1}{362880}$$

$$\int_{\overline{\mathcal{M}}_{3,1}}\lambda_3\lambda_1^2\psi_1^2=\frac{1}{60480}$$

$$\int_{\overline{\mathcal{M}}_{3,1}}\lambda_1\lambda_3\psi_1^3=\frac{41}{145120}$$

6. A simple proof of the Witten Conjecture

6.1. Localization and The Hurwitz Numbers

For any nonnegative integer m, let

$$\mathbb{P}^1[m] = \mathbb{P}^1_{(0)} \cup \mathbb{P}^1_{(1)} \cup \cdots \cup \mathbb{P}^1_{(m)}$$

be a chain of $m+1$ copies $\mathbb{P}^1$, where $\mathbb{P}^1_{(l)}$ is glued to $\mathbb{P}^1_{(l+1)}$ at $p_1^{(l)}$ for each l $(0 \leq l \leq m-1)$. Suppose $p_1^m \neq p_1^{(m-1)}$ a fixed point on $\mathbb{P}^1_{(m)}$.
Let μ be a partition of $d > 0$ and $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$, the virtual dimension $r = 2g - 2 + d + l(\mu)$, be the moduli space of relative stable morphism to $\mathbb{P}^1$. That is, $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$ is the moduli space of morphisms

$$f : (C, x_1, \cdots, x_{l(\mu)}) \longrightarrow (\mathbb{P}^1[m], p_1^{(m)})$$

satisfying some relative stable conditions. The $\mathbb{C}^*$-action on $\mathbb{P}^1$: $t \cdot [z^0, z^1] = [tz^0, z^1]$ induces an action on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$ and $\mathrm{Sym}^r\mathbb{P}^1 = \mathbb{P}^r$. The two fixed points are $q^0 = [0,1]$ and $q^1 = [1,0]$. Under this action, the branching morphism $\mathrm{Br} : \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu) \longrightarrow \mathbb{P}^r$ is $\mathbb{C}^*$-equivariant. The $\mathbb{C}^*$ fixed points in $\mathbb{P}^r$ are given by

$$p_i = (r-i)q^0 + iq^1 = [i, d-i, 0, \cdots, 0], \quad 0 \leq i \leq r.$$

Let H be the first Chern class of $\mathcal{O}_{\mathbb{P}^r}(1)$, then the equivariant cohomology group of $\mathbb{P}^r$ is

$$H^*_{\mathbb{C}^*}(\mathbb{P}^r;\mathbb{Z}) = \frac{\mathbb{Z}[H,u]}{\Pi_{i=1}^r (H-iu)}, \quad H|_{p_i} = iu \in H^2_{\mathbb{C}^*}(\mathbb{P}^r;\mathbb{Z}).$$

Define the Hurwitz numbers

$$H_{g,\mu} = \int_{[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)]^{\mathrm{virt}}} \mathrm{Br}^* H^r = \int_{[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)]^{\mathrm{virt}}} \mathrm{Br}^*(\prod_{k=1}^r (H - w_k u)) \qquad (6.1)$$

with $H \in H^2(\mathbb{P}^r;\mathbb{Z})$ the hyperplane class and for any $w_k \in \mathbb{Z}$ $(1 \leq k \leq r)$. Let $F_i = \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)^{\mathbb{C}^*} \cap \mathrm{Br}^{-1}(p_i)$, then $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)^{\mathbb{C}^*} = \sqcup_{k=0}^r F_k$. Denote

$$\widetilde{I}^k_{g,\mu} = \int_{[F_k]^{\mathrm{virt}}} \frac{u^r}{e_{\mathbb{C}^*}(N^{\mathrm{virt}})},$$

then from the virtual localization formula we have

$$\begin{aligned} H_{g,\mu} &= \sum_{k=0}^r \int_{[F_k]^{\mathrm{virt}}} \frac{\mathrm{Br}^*(\Pi_{l=1}^r (H - w_l u))|_{F_k}}{e_{\mathbb{C}^*}(N^{\mathrm{virt}})} = \sum_{k=0}^r \int_{[F_k]^{\mathrm{virt}}} \frac{\Pi_{l=1}^r (H|_{\mathrm{Br}(F_k)} - w_l u)}{e_{\mathbb{C}^*}(N^{\mathrm{virt}})} \\ &= \sum_{k=0}^r \sum_{l=1}^r (k - w_l) \widetilde{I}^k_{g,\mu}. \end{aligned}$$

Taking

$$(w_1, \cdots, w_r) = (1, 2, \cdots, r)$$

and

$$(w_1, \cdots, w_r) = (0, 2, \cdots, r),$$

we get

$$(-1)^r r! \widetilde{I}^0_{g,\mu} = H_{g,\mu} = (-1)^{r-1}(r-1)! \widetilde{I}^1_{g,\mu}$$

respectively, this relation is equivalent to: ELSV formula and cut-and-join equation:

$$H_{g,\mu} = \frac{r!}{|\mathrm{Aut}(\mu)|} \prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}, \tag{6.2}$$

$$\begin{aligned} H_{g,\mu} &= \sum_{\nu\in J(\mu)} I_1(\nu) H_{g,\nu} + \sum_{\nu \in C(\mu)} I_2(\nu) H_{g-1,\nu} \\ &+ \sum_{g_1+g_2=g} \sum_{\nu^1\cup\nu^2\in C(\mu)} \binom{r-1}{2g_1-2+|\nu^1|+l(\nu^1)} I_3(\nu^1,\nu^2) H_{g_1,\nu^1} H_{g_2,\nu^2}. \end{aligned} \tag{6.3}$$

Define the formal power series

$$\Phi(\lambda, p) = \sum_{\mu} \sum_{g \geq 0} H_{g,\mu} \frac{\lambda^{2g-2+|\mu|+l(\mu)}}{(2g-2+|\mu|+l(\mu))!} p_\mu, \tag{6.4}$$

then we have the following version of cut-and-join equation

$$\frac{\partial \Phi}{\partial \lambda} = \frac{1}{2} \sum_{i,j \geq 1} \left(ijp_{i+j} \frac{\partial^2 \Phi}{\partial p_i \partial p_j} + ijp_{i+j} \frac{\partial \Phi}{\partial p_i} \frac{\partial \Phi}{\partial p_j} + (i+j) p_i p_j \frac{\partial \Phi}{\partial p_{i+j}} \right). \tag{6.5}$$

At last, we define

$$\Phi_{g,n}(z, p) = \sum_{d \geq 1} \sum_{\mu \vdash d, l(\alpha) = n} \frac{H_{g,\mu}}{r!} p_\mu z^d, \tag{6.6}$$

by simple calculation, we can rewrite above formula in the following form

$$\Phi_{g,n}(z; p) = \frac{1}{n!} \sum_{b_1, \cdots, b_n \geq 0, 0 \leq k \leq g} (-1)^k \langle \tau_{b_1} \cdots \tau_{b_n} \lambda_k \rangle_g \prod_{i=1}^{n} \phi_{b_i}(z; p), \tag{6.7}$$

where

$$\phi_i(z; p) = \sum_{m \geq 0} \frac{m^{m+i}}{m!} p_m z^m, \quad i \geq 0. \tag{6.8}$$

6.2. Symmetrization Operator and Rooted Series

In this section, we use the method in [Goulden-Jackson-Vakil(GJV)] to prove the recursion formula which implies the Witten conjecture/Kontsevich theorem. Their method consists of the following steps: (1) introduce three operators to change the variables; (2) compare the leading coefficient of both sides to derive the recursion formula which implies the Witten conjecture. Kim-Liu have proved the Witten conjecture via the asymptotic analysis which writes each $\mu_i = x_i N$ for some $x_i \in \mathbb{R}$ and $N \in \mathbb{N}$. The main problem arising in Kim-Liu is the asymptotic expansion of series

$$e^{-n} \sum_{p+q=n} \frac{p^{p+k+1} q^{q+l+1}}{p!q!}, \quad e^{-n} \sum_{p+q=n}^{n} \frac{p^{p+k+1} q^{q-1}}{p!q!}$$

for any $k, l \in \mathbb{N}$ which are not easy to compute. The idea here is that by using the method in [GJV], we can avoid these problems to derive the recursion formula. First, we symmetrize $\Phi_{g,n}(z,p)$ by using the linear symmetrization operator $\square_n$

$$\square_n(p_\alpha z^{|\alpha|}) = \delta_{l(\alpha),n} \sum_{\sigma \in S_n} x_{\sigma(1)}^{\alpha_1} \cdots x_{\sigma(n)}^{\alpha_n}, \tag{6.9}$$

where S^n is the n-order symmetric group.

It is easy to prove that for $n, g \geq 1$ or $n \geq 3, g \geq 0$ we have

$$\square_n(\Phi_{g,n}(z,p))(x_1, \cdots, x_n) = \frac{1}{n!} \sum_{b_1,\cdots,b_n \geq 0, 0 \leq k \leq g} (-1)^k \langle \tau_{b_1} \cdots \tau_{b_n} \lambda_k \rangle_g \sum_{\sigma \in S_n} \prod_{i=1}^{n} \phi_{b_i}(x_{\sigma(i)}), \tag{6.10}$$

where

$$\phi_i(x) := \phi(x;1) = \sum_{m \geq 1} \frac{m^{m+i}}{m!} x^m. \tag{6.11}$$

the **rooted tree series** $w(x)$ is defined by

$$w(x) = \sum_{m \geq 1} \frac{m^{m-1}}{m!} x^m \Longrightarrow \phi_i(x) = \left(x \frac{d}{dx} \right)^{i+1} w(x) := \nabla_x^{i+1} w(x) \tag{6.12}$$

with $\nabla_x := x\frac{d}{dx}$. The rooted tree series is the unique formal power series solution of the functional equation

$$w(x) = x e^{w(x)}. \tag{6.13}$$

Let $y(x) := \frac{1}{1-w(x)}$ and $y_j = y(x_j)$, we consider change of variables using the operator

$$L : \mathbb{Q}[[x_1, \cdots, x_n]] \longrightarrow \mathbb{Q}[[y_1, \cdots, y_n]], \quad f(x_1, \cdots, x_n) \longmapsto f(y_1, \cdots, y_n). \tag{6.14}$$

$$F_i(y) := [(y^2 - y)\nabla_y]^i (y-1) := \sum_{j=1}^{i+1} f(j,i) y^{2i+2-j}, \tag{6.15}$$

one can show that

$$f(j,i) = -(2i-j)!! \left[1 + \sum_{k=1}^{i-1} \frac{2k+3-j}{(2k+2-j)!!} f(j-1,k) \right], \quad 2 \le j \le i+1, i \ge 1. \tag{6.16}$$

For $i = 1, 2$, it turns to the explicit expression

$$f(1,i) = (2i-1)!!, \quad f(2,i) = -(2i-2)!! \left[1 + \sum_{k=1}^{i-1} \frac{(2k+1)!!}{(2k)!!} \right] = -\frac{(2i+1)!!}{3}. \tag{6.17}$$

6.3. Proof of the DVV Conjecture

For $i, j \ge 0$, $i + j \le n$, let $\underset{i,j}{\overset{x}{\Box}}$ be the mapping, applied to a series in $x_1, \cdots, x_n$, given by

$$\underset{i,j}{\overset{x}{\Box}} f(x_1, \cdots, x_n) = \sum_{\mathcal{R},\mathcal{S},\mathcal{T}} f(x_\mathcal{R}, x_\mathcal{S}, x_\mathcal{T}), \tag{6.18}$$

where the summation is over all ordered partitions $(\mathcal{R}, \mathcal{S}, \mathcal{T})$ of $\{1, \cdots, n\}$, where $\mathcal{R} = \{x_{r_1}, \cdots, x_{r_i}\}$, $\mathcal{S} = \{x_{s_1}, \cdots, x_{s_j}\}$, $\mathcal{T} = \{x_{t_1}, \cdots, x_{t_{n-i-j}}\}$ and

$$(x_\mathcal{R}, x_\mathcal{S}, x_\mathcal{T}) = (x_{r_1}, \cdots, x_{r_i}, x_{s_1}, \cdots, x_{s_j}, x_{t_1}, \cdots, x_{t_{n-i-j}}),$$

The following result gives an expression for the result of applying the symmetrization operator $\Box_n$ to the cut-and-join equation for $\Phi_{g,n}(z,p)$. Denote $\triangle_{y_j} := (y_j^2 - y_j)\nabla_{y_j}$. Applying the symmetrization operator $\Box_n$ to the join-cut Equation, one can easily prove the following version of join-and-cut equation

$$\left(\sum_{i=1}^{n}(y_i-1)\nabla_{y_i}+n+2g-2\right)L\Box_n\Phi_{g,n}(y_1,\cdots,y_n)=T_1'+T_2'+T_3'+T_4', \tag{6.19}$$

where

$$\begin{aligned}
T_1' &= \frac{1}{2}\sum_{i=1}^{n}\left(\triangle_{y_i}\triangle_{y_{n+1}}L\Box_{n+1}\Phi_{g-1,n+1}(y_1,\cdots,y_{n+1})\right)|_{y_{n+1}=y_i},\\
T_2' &= \underset{1,1}{\overset{y}{\Box}} y_1^2\frac{y_2-1}{y_1-y_2}\triangle_{y_1}L\Box_{n-1}\Phi_{g,n-1}(y_1,y_3,\cdots,y_n),\\
T_3' &= \sum_{k=3}^{n}\underset{1,k-1}{\overset{y}{\Box}}\left(\triangle_{y_1}L\Box_k\Phi_{0,k}(y_1,\cdots,y_k)\right)\left(\triangle_{y_1}L\Box_{n-k+1}\Phi_{g,n-k+1}(y_1,y_{k+1},\cdots,y_n)\right),\\
T_4' &= \frac{1}{2}\sum_{\substack{1\le k\le n\\ 1\le a\le g-1}}\underset{1,k-1}{\overset{y}{\Box}}\\
&\cdot\left(\triangle_{y_1}L\Box_k\Phi_{a,k}(y_1,\cdots,y_k)\right)\left(\triangle_{y_1}L\Box_{n-k+1}\Phi_{g-a,n-k+1}(y_1,y_{k+1},\cdots,y_n)\right).
\end{aligned}$$

First we have the following expansion formula

$$L\left(\prod_{i=1}^{n}\phi_{b_i}(x_{\sigma(i)})\right)=\prod_{i=1}^{n}(2b_i-1)!!y_{\sigma(i)}^{2b_i+1}+\text{lower terms.} \tag{6.20}$$

From this point, we see that the polynomial $L\square_n H_n^g(y_1,\cdots,y_n)$ can be written as

$$L\square_n\Phi_{g,n}(y_1,\cdots,y_n)=\sum_{b_1+\cdots+b_n=3g-3+n}\langle\tau_{b_1}\cdots\tau_{b_n}\rangle_g\prod_{i=1}^{n}(2b_i-1)!!y_i^{2b_i+1}+\text{l.t.},$$

where l.t. means lower order terms. We write the left hand side of equation (6.19) by *LHS* while another side by RHS_1, RHS_2, RHS_3 and RHS_4, by simply calculating, we find (where $S=\{b_2,\cdots,b_n\}$)

$$\begin{aligned}
LHS &= (2b_1+1)!!(2b_2-1)!!\cdots(2b_n-1)!!\langle\tau_{b_1}\cdots\tau_{b_n}\rangle_g\\
RHS_1 &= \frac{1}{2}\sum_{a+b=b_1-2}(2a+1)!!(2b+1)!!\prod_{l=2}^{n}(2b_l-1)!!\langle\tau_a\tau_b\tau_{b_2}\cdots\tau_{b_n}\rangle_{g-1}\\
RHS_2 &= \sum_{l=2}^{n}(2(b_1+b_l-1)+1)!!(2b_2-1)!!\cdots(2b_{l-1}-1)!!(2b_{l+1}-1)!!\cdots(2b_n-1)!!\\
&\quad\cdot\ \langle\sigma_{b_1+b_l-1}\sigma_{b_2}\cdots\sigma_{b_{l-1}}\sigma_{b_{l+1}}\cdots\sigma_{b_n}\rangle_g\\
RTS_{3,4} &= \frac{1}{2}\sum_{X\cup Y=S}\sum_{\substack{a+b=b_1-2\\ g_1+g_2=g}}(2a+1)!!(2b+1)!!\prod_{l=2}^{n}(2b_l-1)!!\langle\tau_a\prod_{\alpha\in X}\tau_\alpha\rangle_{g_1}\langle\tau_b\prod_{\beta\in Y}\tau_\beta\rangle_{g_2},
\end{aligned}$$

Finally, multiplying the constant $(2b_2+1)\cdots(2b_n+1)$, we obtain the recursion formula as conjectured by Dijkgraaf-Verlinde-Verlinde which implies the Witten conjecture

$$\langle\widetilde{\tau}_{b_1}\prod_{l=2}^{n}\widetilde{\tau}_{b_l}\rangle_g = \sum_{l=2}^{n}(2b_l+1)\langle\widetilde{\tau}_{b_1+b_l-1}\prod_{k=2,k\neq l}^{n}\widetilde{\tau}_{b_k}\rangle_g + \frac{1}{2}\sum_{a+b=b_1-2}\langle\widetilde{\tau}_a\widetilde{\tau}_b\prod_{l=2}^{n}\widetilde{\tau}_{b_l}\rangle_{g-1}$$
$$\frac{1}{2}\sum_{X\cup Y=\{b_2,\cdots,b_n\}}\sum_{a+b=b_1-2,g_1+g_2=g}\langle\widetilde{\tau}_a\prod_{\alpha\in X}\widetilde{\tau}_\alpha\rangle_{g_1}\langle\widetilde{\tau}_b\prod_{\beta\in Y}\widetilde{\tau}_\beta\rangle_{g_2}.$$

where $\widetilde{\tau}_{b_l}=[(2b_l+1)!!]\tau_{b_l}$.

Considering the "minimum degree" of (6.19), Goulden, Jackson and Vakil gave a short proof of the λ_g conjecture without using the Gromov-Witten theory.

Maybe the degree 0 Virasoro conjecture for surfaces or Faber conjecture can be proved through this method (Working in progress with Lin Chen, Y.S. Kim, C.-C. Liu, Kefeng Liu and Hao Xu):

$$\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\cdots\psi_n^{k_n}\lambda_g\lambda_{g-1}=\frac{(2g+n-3)!(2g-1)!!}{(2g-1)!(2k_1-1)!!\cdots(2k_n-1)!!}\int_{\overline{\mathcal{M}}_{g,1}}\psi_1^{g-1}\lambda_g\lambda_{g-1}.$$

where

$$\int_{\overline{\mathcal{M}}_{g,1}}\psi_1^{g-1}\lambda_g\lambda_{g-1}=\frac{1}{2^{2g-1}(2g-1)!!}\frac{|B_{2g}|}{2g}.$$

Thank You Very Much!